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Finite Math

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2.5 - Exponential Functions

Exponential functions are functions of the form

$$\boxed{f(x) = b^x,\ b > 0,\ b \neq 1}$$

*b is called the base

Why the restrictions on b?

- If $b = 1$, then $f(x) = 1^x = 1$.
- If $b < 0$, then we could end up with, for example, ~~if~~ $b = -1$

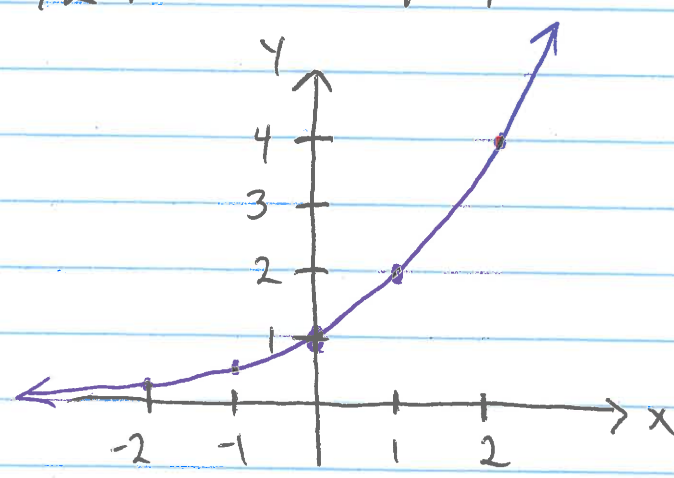
$$f(1/2) = (-1)^{1/2} = i \ (= \sqrt{-1})$$

an imaginary number!

- If $b = 0$, for negative x-values, f is not defined.

Ex: Graph $f(x) = 2^x$.

x	-2	-1	0	1	2
f(x)	1/4	1/2	1	2	4

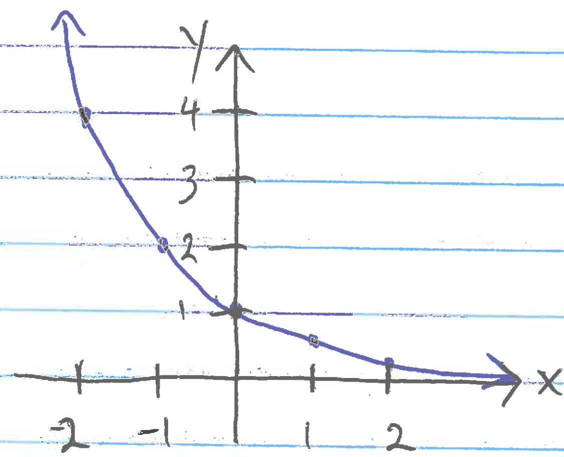


②

The graph of $f(x)=b^x$ looks similar to 2^x for $b>1$, but what about $b<1$?

Ex: Graph $f(x)=\left(\frac{1}{2}\right)^x$.

x	-2	-1	0	1	2
$f(x)$	4	2	1	$1/2$	$1/4$



Notice $\left(\frac{1}{2}\right)^x=(2^{-1})^x=2^{-x}$ so that when $b<1$, $b=\frac{1}{c}$, $c>1$

$f(x)=b^x=\left(\frac{1}{c}\right)^x=c^{-x}$

So we can always keep the base larger than 1 by using a minus sign on the exponent if necessary.

The graph of $f(x)=b^x$, $b>0$, $b\neq 1$ satisfies the following properties

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1) All graphs pass through $(0,1)$

2) All graphs are continuous

3) The x-axis is a horizontal asymptote

4) b^x increases if $b>1$

5) b^x decreases if $0<b<1$

General properties of exponential functions

$a,b>0$, $a,b\neq 1$, x,y real #'s

1) $a^x a^y = a^{x+y}$, $\frac{a^x}{a^y} = a^{x-y}$

$(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$, $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

2) $a^x = a^y$ if and only if $x=y$

3) $a^x = b^x$ if and only if $a=b$
$(x\neq 0)$

A Special Number: e

There is one number that occurs in applications a lot: the natural number e. One definition of e is the value which:

$$\left(1+\frac{1}{x}\right)^x$$

approaches as x tends towards ∞.

This number often shows up in growth and decay models. If c is the initial amount and r is the growth/decay rate, then the amount after time t is

$$\boxed{A = ce^{rt}}$$

$r>0$ growth
$r<0$ decay

Ex: In 2013, the estimated world population was 7.1 billion people with a relative growth rate of 1.1%.

(a) Write a function modeling the world population t years after 2013.

(b) What is the expected population in 2015? 2025? 2035?

Sol: (a) $c = 7.1$ $r = 1.1\% = 0.011$

t = years after 2013

Population $= P(t) = 7.1e^{0.011t}$ billion.

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(b) 2015 : $t=2$

$P(2)=7.1e^{0.011(2)}=7.1e^{0.022}$
≈ 7.26 billion

2025 : $t=12$

$P(12)=7.1e^{0.011(12)}=7.1e^{0.132}$
≈ 8.1 billion

2035 : $t=22$

$P(22)=7.1e^{0.011(22)}=7.1e^{0.242}$
≈ 9.04 billion

2.6 - Logarithmic Functions

Inverse Functions

The inverse of a function is given by running the function backwards. But when can we do this?

Consider the function $f(x)=x^2$. If we run f backwards on the value 1, what x-value do we get?

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Since $(1)^2 = 1$ & $(-1)^2 = 1$, we get two values when we run x^2 backward! So x^2 is not invertible!

To get the inverse of a function we need each range value to come from exactly one domain value. We call such functions one-to-one.

If we have a one-to-one function

$$y = f(x)$$

we can form the inverse function by switching x&y and solving for y:

$$\boxed{x = f(y) \xrightarrow[y]{\text{solve for}} y = f^{-1}(x)}$$

There is one particular inverse function which will be most useful to us: the inverse of $f(x) = b^x$ $(b>0,\ b\neq 1)$.

Defn: The logarithm of base b is defined as the inverse of b^x, i.e.,

if $y = b^x$, then $x = \log_b y$

Notice that since the domain and range switch when taking inverses, we have

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fcn	domain	range
$f(x)=b^x$	$(-\infty,\infty)$	$(0,\infty)$
$f(x)=\log_b x$	$(0,\infty)$	$(-\infty,\infty)$

Since logarithms are inverse to exponential functions, we get some convenient properties!

$$\boxed{b, M, N > 0,\ b \neq 1,\ p, x \text{ real \#'s}}$$

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b \frac{M}{N} = \log_b M - \log_b N$
7. $\log_b M^p = p \log_b M$
8. $\log_b M = \log_b N$ if and only if $M = N$

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Just as with exponential functions, if we choose our base to be the number e, we get a special logarithm: the <u>natural logarithm</u>

$$\boxed{\log_e x = \ln x}$$

We can actually rewrite a logarithm in any base in terms of ln:

$$\boxed{\log_b x = \frac{\ln x}{\ln b}}$$

** If $y = \log_b x$, then $x = b^y$. Take ln of both sides:

$$\ln x = \ln b^y = y \ln b$$

$$\Rightarrow \quad y = \frac{\ln x}{\ln b} \quad **$$

Note*: this could really be done with any base of logarithm, not just ln

Recall again that exponential growth/decay models are of the form

$$\boxed{A = ce^{rt}}$$

⑨

With logarithms, we can access "r" (the rate of growth/decay) and "t" (the time elapsed).

Ex: The isotope carbon-14 has a half-life of 5730 years.

(a) At what rate does carbon-14 decay?

(b) How long would it take for 90% of a chunk of carbon-14 to decay?

Sol:

(a) Inital mass $= M_0$

Want half of initial mass, so

$$\frac{\cancel{M_0}}{2} = \cancel{M_0} e^{r(5730)} \iff \frac{1}{2} = e^{5730r}$$

Take ln:

$$-\ln 2 = \ln \frac{1}{2} = \ln e^{5730r} = 5730r$$

$$\Rightarrow r = \frac{-\ln 2}{5730} \approx -0.00012$$

So carbon-14 loses 0.12% of its mass per year.

(b) If the mass of M_0 loses 90% of its mass, we're looking for the time it takes for $0.1 M_0$ to remain. So,

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$$0.1\cancel{M_0} = \cancel{M_0} = e^{-0.00012t}$$

$$\Rightarrow \frac{1}{10} = e^{-0.00012t}$$

$$\Rightarrow \ln\frac{1}{10} = -\ln 10 = \ln e^{-0.00012t} = -0.00012t$$

$$\Rightarrow t = \frac{\ln 10}{0.00012} = 19188.21$$

It would take ~19,188.21 years for 90% of the original mass to decay.